SOME CAUCHY-BUNYAKOVSKY-SCHWARZ TYPE INEQUALITIES FOR SEQUENCES OF OPERATORS IN HILBERT SPACES

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Abstract. Some inequalities of Cauchy-Bunyakovsky-Schwarz type for sequences of bounded linear operators in Hilbert spaces and some applications are given.

1. Introduction

Let $(H;(\cdot,\cdot))$ be a real or complex Hilbert space and $B(H)$ the Banach algebra of all bounded linear operators that map H into H.

We recall that a self-adjoint operator $A \in B(H)$ is positive in $B(H)$ iff $(Ax,x) \geq 0$ for any $x \in H$. The binary relation $A \geq B$ iff $A-B$ is a positive self-adjoint operator, is an *order relation* on $B(H)$. We remark that for any $A \in B(H)$ the operators $U := AA^*$ and $V := A^*A$ are positive self adjoint operators on H and $\|U\| = \|V\| = \|A\|^2$.

In [1], the author has proved the following inequality of Cauchy-Bunyakovsky-Schwarz type in the order of $B(H)$.

Theorem 1. *Let $A_1,\dots,A_n \in B(H)$ and $z_1,\dots,z_n \in \mathbb{K}$ $(\mathbb{R},\mathbb{C})$. Then the following inequality holds:*

$$\sum_{i=1}^{n}|z_i|^2\sum_{i=1}^{n}A_iA_i^* \geq \left(\sum_{i=1}^{n}z_iA_i\right)\left(\sum_{i=1}^{n}\overline{z_i}A_i^*\right) \geq 0. \tag{1.1}$$

Proof. For the sake of completeness, we give here a simple proof of this inequality.

For any $i,j \in \{1,\dots,n\}$ one has in the order of $B(H)$:

$$\left(\overline{z_i}A_j - \overline{z_j}A_i\right)\left(\overline{z_i}A_j - \overline{z_j}A_i\right)^* \geq 0,$$

that is,

$$\left(\overline{z_i}A_j - \overline{z_j}A_i\right)\left(z_iA_j^* - z_jA_i^*\right) \geq 0,$$

from where results

$$|z_i|^2 A_jA_j^* + |z_j|^2 A_iA_i^* \geq \overline{z_i}z_jA_jA_i^* + \overline{z_j}z_iA_iA_j^* \tag{1.2}$$

for any $i,j \in \{1,\dots,n\}$.

If we sum (1.2) over i from 1 to n we deduce

$$\left(\sum_{i=1}^{n}|z_i|^2\right)A_jA_j^* + |z_j|^2\left(\sum_{i=1}^{n}A_iA_i^*\right) \geq z_jA_j\left(\sum_{i=1}^{n}\overline{z_i}A_i^*\right) + \left(\sum_{i=1}^{n}z_iA_i\right)\overline{z_j}A_j^*, \tag{1.3}$$

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for any $j \in \{1, \dots, n\}$.

If we sum (1.3) over j from 1 to n, we deduce

$$\sum_{i=1}^{n} |z_i|^2 \sum_{j=1}^{n} A_j A_j^* + \sum_{j=1}^{n} |z_j|^2 \left(\sum_{i=1}^{n} A_i A_i^* \right) \geq \sum_{j=1}^{n} z_j A_j \sum_{i=1}^{n} \overline{z_i} A_i^* + \left(\sum_{i=1}^{n} z_i A_i \right) \left(\sum_{j=1}^{n} \overline{z_j} A_j^* \right), \tag{1.4}$$

that is,

$$\sum_{k=1}^{n} |z_k|^2 \sum_{k=1}^{n} A_k A_k^* \geq \sum_{k=1}^{n} z_k A_k \sum_{k=1}^{n} \overline{z_k} A_k^* = \left(\sum_{k=1}^{n} z_k A_k \right) \left(\sum_{k=1}^{n} z_k A_k \right)^* \geq 0, \tag{1.5}$$

and the theorem is proved. ∎

The following version of the Cauchy-Bunyakovsky-Schwarz inequality for norms also holds [1].

Corollary 1. *With the assumptions in Theorem 1, one has*

$$\sum_{k=1}^{n} |z_k|^2 \left\| \sum_{k=1}^{n} A_k A_k^* \right\| \geq \left\| \sum_{k=1}^{n} z_k A_k \right\|^2 . \tag{1.6}$$

Proof. The operators:

$$A := \sum_{k=1}^{n} |z_k|^2 \sum_{k=1}^{n} A_k A_k^*, \quad B := \left(\sum_{k=1}^{n} z_k A_k \right) \left(\sum_{k=1}^{n} \overline{z_k} A_k \right)^*$$

are obviously self-adjoint, positive and by (1.1), $A \geq B \geq 0$. Thus $\|A\| \geq \|B\|$ and since,

$$\|A\| = \sum_{k=1}^{n} |z_k|^2 \left\| \sum_{k=1}^{n} A_k A_k^* \right\|$$

and

$$\|B\| = \left\| \sum_{k=1}^{n} z_k A_k \right\|^2$$

the corollary is proved. ∎

For other related results, see [2].

The main aim of this paper is to point out other inequalities similar to (1.6).

2. Norm Inequalities

The following result holds.

Theorem 2. *Let $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ and $A_1, \dots, A_n \in B(H)$. Then one has the inequalities:*

$$
\left\| \sum_{i=1}^{n} \alpha_i A_i \right\|^2 \leq \begin{cases} \max\limits_{i=\overline{1,n}} |\alpha_i|^2 \sum\limits_{i=1}^{n} \|A_i\|^2 \\ \\ \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2p} \right)^{\frac{1}{p}} \left(\sum\limits_{i=1}^{n} \|A_i\|^{2q} \right)^{\frac{1}{q}} \quad \text{if } \ p>1, \ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sum\limits_{i=1}^{n} |\alpha_i|^2 \max\limits_{i=\overline{1,n}} \|A_i\|^2 \end{cases}
$$
$$
+ \begin{cases} \max\limits_{1\leq i\neq j\leq n} \{|\alpha_i|\,|\alpha_j|\} \sum\limits_{1\leq i\neq j\leq n} \left\|A_i A_j^*\right\| \\ \\ \left[\left(\sum\limits_{i=1}^{n} |\alpha_i|^r \right)^2 - \left(\sum\limits_{i=1}^{n} |\alpha_i|^{2r} \right)^{\frac{1}{2}} \right] \left(\sum\limits_{1\leq i\neq j\leq n} \left\|A_i A_j^*\right\|^s \right)^{\frac{1}{s}} \\ \text{if } \ r>1, \ \frac{1}{r}+\frac{1}{s}=1; \\ \left[\left(\sum\limits_{i=1}^{n} |\alpha_i| \right)^2 - \left(\sum\limits_{i=1}^{n} |\alpha_i|^2 \right) \right] \max\limits_{1\leq i\neq j\leq n} \left\|A_i A_j^*\right\|, \end{cases} \tag{2.1}
$$

where (2.1) should be seen as all the 9 possible configurations.

Proof. We have

$$
\begin{aligned} 0 &\leq \left(\sum_{i=1}^{n} \alpha_i A_i \right) \left(\sum_{i=1}^{n} \alpha_i A_i \right)^* = \left(\sum_{i=1}^{n} \alpha_i A_i \right) \left(\sum_{j=1}^{n} \overline{\alpha_j} A_j^* \right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} A_i A_j^* = \sum_{i=1}^{n} |\alpha_i|^2 A_i A_i^* + \sum_{1\leq i\neq j\leq n} \alpha_i \overline{\alpha_j} A_i A_j^*. \end{aligned} \tag{2.2}
$$

Taking the norm in (2.2) and observing that $\|UU^*\| = \|U\|^2$ for any $U \in B(H)$, one has the inequality

$$
\begin{aligned} \left\| \sum_{i=1}^{n} \alpha_i A_i \right\|^2 &= \left\| \sum_{i=1}^{n} |\alpha_i|^2 A_i A_i^* + \sum_{1\leq i\neq j\leq n} \alpha_i \overline{\alpha_j} A_i A_j^* \right\| \\ &\leq \sum_{i=1}^{n} |\alpha_i|^2 \|A_i A_i^*\| + \sum_{1\leq i\neq j\leq n} |\alpha_i|\,|\alpha_j| \left\|A_i A_j^*\right\| \\ &= \sum_{i=1}^{n} |\alpha_i|^2 \|A_i\|^2 + \sum_{1\leq i\neq j\leq n} |\alpha_i|\,|\alpha_j| \left\|A_i A_j^*\right\|. \end{aligned} \tag{2.3}
$$

Using Hölder's inequality, we may write that:

$$\sum_{i=1}^{n}|\alpha_i|^2\|A_i\|^2 \leq \begin{cases} \max_{i=\overline{1,n}}|\alpha_i|^2 \sum_{1\leq i\neq j\leq n}\|A_i\|^2 \\ \\ \left(\sum_{i=1}^{n}|\alpha_i|^{2p}\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}\|A_i\|^{2q}\right)^{\frac{1}{q}} & \text{if } \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \sum_{i=1}^{n}|\alpha_i|^2 \max_{i=\overline{1,n}}\|A_i\|^2. \end{cases} \tag{2.4}$$

Also, Hölder's inequality for double sums produces

(2.5)

$$\sum_{1\leq i\neq j\leq n}|\alpha_i||\alpha_j|\left\|A_iA_j^*\right\| \leq \begin{cases} \max_{1\leq i\neq j\leq n}\{|\alpha_i||\alpha_j|\}\sum_{1\leq i\neq j\leq n}\left\|A_iA_j^*\right\| \\ \\ \left(\sum_{1\leq i\neq j\leq n}|\alpha_i|^r|\alpha_j|^r\right)^{\frac{1}{r}}\left(\sum_{1\leq i\neq j\leq n}\left\|A_iA_j^*\right\|^s\right)^{\frac{1}{s}} \\ \text{if } \ r>1,\ \frac{1}{r}+\frac{1}{s}=1; \\ \sum_{1\leq i\neq j\leq n}|\alpha_i||\alpha_j|\max_{1\leq i\neq j\leq n}\left\|A_iA_j^*\right\|, \end{cases}$$

$$= \begin{cases} \max_{1\leq i\neq j\leq n}\{|\alpha_i||\alpha_j|\}\sum_{1\leq i\neq j\leq n}\left\|A_iA_j^*\right\| \\ \\ \left[\left(\sum_{i=1}^{n}|\alpha_i|^r\right)^2-\left(\sum_{i=1}^{n}|\alpha_i|^{2r}\right)^2\right]^{\frac{1}{2}}\left(\sum_{i=1}^{n}\left\|A_iA_j^*\right\|^s\right)^{\frac{1}{s}} \\ \text{if } \ r>1,\ \frac{1}{r}+\frac{1}{s}=1; \\ \left[\left(\sum_{i=1}^{n}|\alpha_i|\right)^2-\left(\sum_{i=1}^{n}|\alpha_i|^2\right)\right]\max_{1\leq i\neq j\leq n}\left\|A_iA_j^*\right\|, \end{cases}$$

Using (2.3) and (2.4), (2.5) one deduces the desired inequality (2.1). ∎

The following corollaries are natural consequences.

Corollary 2. *With the assumptions of Theorem 2, one has the inequality*

$$\left\|\sum_{i=1}^{n}\alpha_iA_i\right\| \leq \max_{i=\overline{1,n}}|\alpha_i|\left(\sum_{i,j=1}^{n}\left\|A_iA_j^*\right\|\right)^{\frac{1}{2}}. \tag{2.6}$$

Proof. Follows by the first line in (2.1) on taking into account that

$$\max_{1\leq i\neq j\leq n}\{|\alpha_i||\alpha_j|\} \leq \max_{i=\overline{1,n}}|\alpha_i|^2,$$

and

$$\sum_{i,j=1}^{n}\left\|A_iA_j^*\right\| = \sum_{i=1}^{n}\|A_i\|^2+\sum_{1\leq i\neq j\leq n}\left\|A_iA_j^*\right\|.$$

∎

Corollary 3. *With the assumptions in Theorem 2, one has the inequality:*

$$\left\|\sum_{i=1}^{n}\alpha_i A_i\right\|^2 \leq \left(\sum_{i=1}^{n}|\alpha_i|^{2p}\right)^{\frac{1}{p}}\left[\left(\sum_{i=1}^{n}\|A_i\|^{2q}\right)^{\frac{1}{q}}+(n-1)\left(\sum_{1\leq i\neq j\leq n}\left\|A_iA_j^*\right\|^q\right)^{\frac{1}{q}}\right], \tag{2.7}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using the Cauchy-Bunyakovsky-Schwarz inequality for positive numbers

$$\left(\sum_{i=1}^{n}a_i\right)^2 \leq n\sum_{i=1}^{n}a_i^2$$

we may write that

$$\begin{aligned}\left(\sum_{i=1}^{n}|\alpha_i|^p\right)^2-\sum_{i=1}^{n}|\alpha_i|^{2p} &\leq n\sum_{i=1}^{n}|\alpha_i|^{2p}-\sum_{i=1}^{n}|\alpha_i|^{2p}\\ &=(n-1)\sum_{i=1}^{n}|\alpha_i|^{2p}.\end{aligned}$$

Now, using the second line in (2.1) for $r=p$, $s=q$, we deduce the desired result (2.7). ∎

Corollary 4. *With the assumptions in Theorem 2, one has the inequality*

$$\left\|\sum_{i=1}^{n}\alpha_i A_i\right\|^2 \leq \sum_{i=1}^{n}|\alpha_i|^2\left[\max_{i=\overline{1,n}}\|A_i\|^2+(n-1)\max_{1\leq i\neq j\leq n}\left\|A_iA_j^*\right\|\right]. \tag{2.8}$$

Proof. Follows by the third line of (2.1) on taking into account that

$$\left[\left(\sum_{i=1}^{n}|\alpha_i|\right)^2-\sum_{i=1}^{n}|\alpha_i|^2\right]^{\frac{1}{2}} \leq (n-1)\sum_{i=1}^{n}|\alpha_i|^2.$$

∎

Another interesting particular case is embodied in the following corollary as well.

Corollary 5. *With the assumptions in Theorem 2, one has the inequality*

$$\left\|\sum_{i=1}^{n}\alpha_i A_i\right\|^2 \leq \sum_{i=1}^{n}|\alpha_i|^2\left[\max_{i=\overline{1,n}}\|A_i\|^2+\left(\sum_{1\leq i\neq j\leq n}\left\|A_iA_j^*\right\|^2\right)^{\frac{1}{2}}\right]. \tag{2.9}$$

Proof. It is obvious that

$$\left[\left(\sum_{i=1}^{n}|\alpha_i|^2\right)^2-\sum_{i=1}^{n}|\alpha_i|^4\right]^{\frac{1}{2}} \leq \sum_{i=1}^{n}|\alpha_i|^2.$$

Thus, combining the third line in the first bracket in (2.1) with the second line for $r=s=2$ in the second bracket, the inequality (2.9) is obtained. ∎

Remark 1. *If one is interested in obtaining bounds in terms of* $\sum_{i=1}^{n}|\alpha_i|^2$, *there are other possibilities as shown below. Obviously, since*

$$\max_{1\leq i\neq j\leq n}\{|\alpha_i|\,|\alpha_j|\}\leq \max_{i=\overline{1,n}}|\alpha_i|^2\leq \sum_{i=1}^{n}|\alpha_i|^2.$$

then, by (2.1), in choosing the third line in the first bracket with the first line in the second bracket, one would obtain

$$\left\|\sum_{i=1}^{n}\alpha_i A_i\right\|^2\leq \sum_{i=1}^{n}|\alpha_i|^2\left[\max_{i=\overline{1,n}}\|A_i\|^2+\sum_{1\leq i\neq j\leq n}\left\|A_iA_j^*\right\|\right]. \tag{2.10}$$

Also, it is evident that

$$\left[\left(\sum_{i=1}^{n}|\alpha_i|^r\right)^2-\left(\sum_{i=1}^{n}|\alpha_i|^{2r}\right)^{\frac{1}{2}}\right]^{\frac{1}{r}}\leq\left(\sum_{i=1}^{n}|\alpha_i|^r\right)^{\frac{2}{r}}.$$

By the monotonicity of the power mean $\left(\frac{1}{n}\sum_{i=1}^{n}a_i^m\right)^{\frac{1}{m}}$ *as a function of* $m\in\mathbb{R}$, *we have*

$$\left(\frac{\sum_{i=1}^{n}|\alpha_i|^r}{n}\right)^{\frac{1}{r}}\leq\left(\frac{\sum_{i=1}^{n}|\alpha_i|^2}{n}\right)^{\frac{1}{2}},\qquad 1<r\leq 2,$$

giving

$$\left(\sum_{i=1}^{n}|\alpha_i|^r\right)^{\frac{2}{r}}\leq n^{\frac{2}{r}-1}\sum_{i=1}^{n}|\alpha_i|^2.$$

Thus, using the third line in the first bracket of (2.1) combined with the second line in the second bracket for $1\leq r\leq 2$, $\frac{1}{s}+\frac{1}{r}=1$, *we deduce*

$$\left\|\sum_{i=1}^{n}\alpha_i A_i\right\|^2\leq \sum_{i=1}^{n}|\alpha_i|^2\left[\max_{i=\overline{1,n}}\|A_i\|^2+n^{\frac{2}{r}-1}\left(\sum_{1\leq i\neq j\leq n}\left\|A_iA_j^*\right\|^s\right)^{\frac{1}{s}}\right]. \tag{2.11}$$

Note that for $r=s=2$, *we recapture (2.9).*

The following particular result also holds.

Proposition 1. *Let* $\alpha_1,\dots,\alpha_n\in\mathbb{K}$ *and* $A_1,\dots,A_n\in B(H)$ *with the property that* $A_iA_j^*=0$ *for any* $i\neq j$, $i,j\in\{1,\dots,n\}$. *Then one has the inequality;*

$$\left\|\sum_{i=1}^{n}\alpha_i A_i\right\|\leq\begin{cases}\max\limits_{i=\overline{1,n}}|\alpha_i|\left(\sum\limits_{i=1}^{n}\|A_i\|^2\right)^{\frac{1}{2}},\\[2ex] \left(\sum\limits_{i=1}^{n}|\alpha_i|^{2p}\right)^{\frac{1}{2p}}\left(\sum\limits_{i=1}^{n}\|A_i\|^{2q}\right)^{\frac{1}{2q}} & \text{if }\ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \left(\sum\limits_{i=1}^{n}|\alpha_i|^2\right)^{\frac{1}{2}}\max\limits_{i=\overline{1,n}}\|A_i\|.\end{cases} \tag{2.12}$$

If by $M(\boldsymbol{\alpha},\mathbf{A})$ we denote any of the bounds provided by (2.1), (2.6), (2.7), (2.8), (2.9), (2.10) or (2.11), then we may state the following proposition as well.

Proposition 2. *Under the assumptions of Theorem 2, we have:*

(i) *For any $x \in H$*

$$\left\| \sum_{i=1}^{n} \alpha_i A_i x \right\|^2 \leq \|x\|^2 M(\boldsymbol{\alpha}, \mathbf{A}). \tag{2.13}$$

(ii) *For any $x, y \in H$,*

$$\left| \sum_{i=1}^{n} \alpha_i \langle A_i x, y \rangle \right|^2 \leq \|x\|^2 \|y\|^2 M(\boldsymbol{\alpha}, \mathbf{A}). \tag{2.14}$$

Proof. (i) Obviously,

$$\begin{aligned} \left\| \sum_{i=1}^{n} \alpha_i A_i x \right\|^2 &= \left\| \left(\sum_{i=1}^{n} \alpha_i A_i \right)(x) \right\|^2 \leq \left\| \sum_{i=1}^{n} \alpha_i A_i \right\|^2 \|x\|^2 \\ &\leq M(\boldsymbol{\alpha}, \mathbf{A}) \|x\|^2. \end{aligned}$$

(ii) We have

$$\left| \sum_{i=1}^{n} \alpha_i \langle A_i x, y \rangle \right|^2 = \left| \left\langle \sum_{i=1}^{n} \alpha_i A_i x, y \right\rangle \right|^2 = \left\| \sum_{i=1}^{n} \alpha_i A_i x \right\|^2 \|y\|^2,$$

which, by (i), gives the desired result (2.14).

∎

3. Inequalities for Vectors in Hilbert Spaces

We consider the non zero vectors $y_1, \dots, y_n \in H$. Define the operators

$$A_i : H \to H, \quad A_i x = \frac{(x, y_i)}{\|y_i\|} \cdot y_i, \quad i \in \{1, \dots, n\}.$$

Since

$$\|A_i\| = \sup_{\|x\|=1} \|A_i x\| = \sup_{\|x\|=1} |(x, y_i)| = \|y_i\|, \quad i \in \{1, \dots, n\} \tag{3.1}$$

then A_i are bounded linear operators in H. Also, since

$$(A_i x, x) = \left(\frac{(x, y_i) y_i}{\|y_i\|}, x \right) = \frac{|(x, y_i)|^2}{\|y_i\|} \geq 0, \quad x \in H, \quad i \in \{1, \dots, n\} \tag{3.2}$$

and

$$\begin{aligned} (A_i x, z) &= \left(\frac{(x, y_i) y_i}{\|y_i\|}, z \right) = \frac{(x, y_i)(y_i, z)}{\|y_i\|}, \\ (x, A_i z) &= \left(x, \frac{(z, y_i) y_i}{\|y_i\|} \right) = \frac{(x, y_i)\overline{(z, y_i)}}{\|y_i\|} = \frac{(x, y_i)(y_i, z)}{\|y_i\|}, \end{aligned}$$

giving

$$(A_i x, z) = (x, A_i z), \quad x, z \in H, \quad i \in \{1, \dots, n\}, \tag{3.3}$$

we may conclude that A_i $(i = 1, \dots, n)$ are positive self-adjoint operators on H.

Since, for any $x \in H$, one has

$$\begin{aligned} \|(A_iA_j)(x)\| &= \|(A_i)(A_jx)\| = \left\| A_i\left(\frac{(x,y_j)\,y_j}{\|y_j\|}\right)\right\| \\ &= \frac{|(x,y_j)|}{\|y_j\|}\|A_iy_j\| = \frac{|(x,y_j)|}{\|y_j\|}\cdot\frac{|(y_j,y_i)|\,\|y_j\|}{\|y_i\|} \\ &= \frac{|(x,y_j)|\,|(y_j,y_i)|}{\|y_i\|},\quad i,j\in\{1,\dots,n\}, \end{aligned}$$

we deduce that

$$\|A_iA_j\| = \sup_{\|\alpha\|=1}\frac{|(x,y_j)|\,|(y_j,y_i)|}{\|y_i\|} = |(y_i,y_j)|;\quad i,j\in\{1,\dots,n\}. \tag{3.4}$$

If $(y_i)_{i=\overline{1,n}}$ is an orthogonal family on H, then $\|A_i\| = 1$ and $A_iA_j = 0$ for $i,j \in \{1,\dots,n\}$, $i \neq j$.

The following inequality for vectors holds.

Theorem 3. *Let $x, y_1, \dots, y_n \in H$ and $\alpha_n, \dots, \alpha_n \in \mathbb{K}$. Then one has the inequalities:*

$$\begin{aligned} &\left\|\sum_{i=1}^n \alpha_i\frac{(x,y_i)}{\|y_i\|}y_i\right\|^2 \\ &\le \|x\|^2\times\begin{cases} \max\limits_{i=\overline{1,n}}|\alpha_i|^2\sum\limits_{i=1}^n\|y_i\|^2 \\ \left(\sum\limits_{i=1}^n|\alpha_i|^{2p}\right)^{\frac1p}\left(\sum\limits_{i=1}^n\|y_i\|^{2q}\right)^{\frac1q} & \text{if } p>1,\ \frac1p+\frac1q=1; \\ \sum\limits_{i=1}^n|\alpha_i|^2\max\limits_{i=\overline{1,n}}\|y_i\|^2 \end{cases} \\ &\quad + \|x\|^2\times\begin{cases} \max\limits_{1\le i\neq j\le n}\{|\alpha_i|\,|\alpha_j|\}\sum\limits_{1\le i\neq j\le n}|(y_i,y_j)| \\ \left[\left(\sum\limits_{i=1}^n|\alpha_i|^r\right)^2-\sum\limits_{i=1}^n|\alpha_i|^{2r}\right]^r\left(\sum\limits_{1\le i\neq j\le n}|(y_i,y_j)|^s\right)^{\frac1s} \\ \text{if } r>1,\ \frac1r+\frac1s=1; \\ \left[\left(\sum\limits_{i=1}^n|\alpha_i|\right)^2-\sum\limits_{i=1}^n|\alpha_i|^2\right]\max\limits_{1\le i\neq j\le n}|(y_i,y_j)|. \end{cases} \end{aligned} \tag{3.5}$$

Proof. Follows by Theorem 2 and Proposition 2, (i) on choosing $A_i = \frac{(\cdot,y_i)}{\|y_i\|}y_i$ and taking into account that $\|A_i\| = \|y_i\|$,

$$\left\|A_iA_j^*\right\| = |(y_i,y_j)|,\qquad i,j\in\{1,\dots,n\}.$$

We omit the details. ∎

Using Corollaries 2–5 and Remark 1, we may state the following particular inequalities:

$$\left\| \sum_{i=1}^{n} \alpha_i \frac{(x, y_i)}{\|y_i\|} y_i \right\| \leq \|x\| \max_{i=\overline{1,n}} |\alpha_i| \left(\sum_{i,j=1}^{n} |(y_i, y_j)| \right)^{\frac{1}{2}}; \tag{3.6}$$

$$\left\| \sum_{i=1}^{n} \alpha_i \frac{(x, y_i)}{\|y_i\|} y_i \right\|^2 \tag{3.7}$$
$$\leq \|x\|^2 \left[\left(\sum_{i=1}^{n} |\alpha_i|^{2p} \right)^{\frac{1}{p}} \left[\left(\sum_{i=1}^{n} \|y_i\|^{2q} \right)^{\frac{1}{q}} + (n-1) \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^q \right)^{\frac{1}{q}} \right] \right],$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$;

$$\left\| \sum_{i=1}^{n} \alpha_i \frac{(x, y_i)}{\|y_i\|} y_i \right\|^2 \leq \|x\|^2 \sum_{i=1}^{n} |\alpha_i|^2 \left[\max_{i=\overline{1,n}} \|y_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |(y_i, y_j)| \right]; \tag{3.8}$$

$$\left\| \sum_{i=1}^{n} \alpha_i \frac{(x, y_i)}{\|y_i\|} y_i \right\|^2 \leq \|x\|^2 \sum_{i=1}^{n} |\alpha_i|^2 \left[\max_{i=\overline{1,n}} \|y_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^2 \right)^{\frac{1}{2}} \right]; \tag{3.9}$$

$$\left\| \sum_{i=1}^{n} \alpha_i \frac{(x, y_i)}{\|y_i\|} y_i \right\|^2 \leq \|x\|^2 \sum_{i=1}^{n} |\alpha_i|^2 \left[\max_{i=\overline{1,n}} \|y_i\|^2 + \sum_{1 \leq i \neq j \leq n} |(y_i, y_j)| \right]; \tag{3.10}$$

$$\left\| \sum_{i=1}^{n} \alpha_i \frac{(x, y_i)}{\|y_i\|} y_i \right\|^2 \tag{3.11}$$
$$\leq \|x\|^2 \sum_{i=1}^{n} |\alpha_i|^2 \left[\max_{i=\overline{1,n}} \|y_i\|^2 + n^{\frac{2}{r}-1} \left(\sum_{1 \leq i \neq j \leq n} |(y_i, y_j)|^s \right)^{\frac{1}{s}} \right],$$

where $1 < r \leq 2$, $\frac{1}{s} + \frac{1}{r} = 1$.

Remark 2. *The choice $\alpha_i = \|y_i\|$ $(i = 1, \ldots, n)$ will produce some interesting bounds for*

$$\left\| \sum_{i=1}^{n} (x, y_i)\, y_i \right\|^2.$$

We omit the details.

References

[1] S.S. Dragomir, On some operatorial inequalities in Hilbert spaces, *Bull. Inst. Pol. Cluj-Napoca* (Romania), **36**(1987), 23-28.

[2] S.S. Dragomir and B. Mond, On Cauchy-Buniakowski-Schwarz's inequality for sequences of linear operators in Hilbert space, *Huston J. Math.*, **22**(1996), No. 2, 329-339.

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